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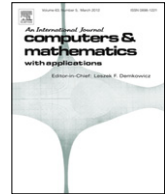
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Non-conforming Galerkin finite element methods for local absorbing boundary conditions of higher order

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ABSTRACT

A new non-conforming finite element discretization methodology for second order elliptic partial differential equations involving higher order local absorbing boundary conditions in 2D and 3D is proposed. The novelty of the approach lies in the application of C^0 -continuous finite element spaces, which is the standard discretization of second order operators, to the discretization of boundary differential operators of order four and higher. For each of these boundary operators, additional terms appear on the boundary nodes in 2D and on the boundary edges in 3D, similarly to interior penalty discontinuous Galerkin methods, which leads to a stable and consistent formulation. In this way, no auxiliary variables on the boundary have to be introduced and trial and test functions of higher smoothness along the boundary are not required. As a consequence, the method leads to lower computational costs for discretizations with higher order elements and is easily integrated in high-order finite element libraries. *A priori* h -convergence error estimates show that the method does not reduce the order of convergence compared to usual Dirichlet, Neumann or Robin boundary conditions if the polynomial degree on the boundary is increased simultaneously. A series of numerical experiments illustrates the utility of the method and validates the theoretical convergence results.

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1. Introduction

Modelling of complex systems in science and engineering, for example in electromagnetics, mechanics, acoustics or quantum mechanics, often require the problem to be reduced to a domain of interest and conditions on its boundary are described to incorporate the state of the system exterior to this computational domain. Very often local absorbing and impedance boundary conditions [1–3] are used, where Dirichlet, Neumann and Robin boundary conditions are the most prominent examples. To achieve higher accuracy, local absorbing boundary conditions of higher order are derived, which possess higher tangential derivatives that are even of order four and higher. In this paper, solutions of second order partial differential equations (PDEs) in a connected Lipschitz domain $\Omega \in \mathbb{R}^d$, $d = 2, 3$ subject to local absorbing boundary conditions (ABCs) on a closed subset Γ of the boundary $\partial\Omega$ are considered. As prominent exponent are the *symmetric local*

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absorbing boundary conditions (see [4, Eq. 3.14] and [5]), which are given for $d = 2$ as

$$\partial_\nu u + \sum_{j=0}^J (-1)^j \partial_\Gamma^j (\alpha_j \partial_\Gamma^j u) = g, \quad (1)$$

where ∂_ν and ∂_Γ denote the normal and tangential derivatives on Γ , $\partial_\nu := \nu \cdot \nabla$, $\partial_\Gamma = \tau \cdot \nabla$, ν is the outer unit normal vector on Γ and τ the unit tangential vector. Furthermore, α_j , $j = 0, \dots, J$ and g are smooth enough functions on Γ , and $J \in \mathbb{N}_0 \cup \{-1\}$ is the order of highest (second) derivatives. Local ABCs for $J = -1, 0$ are well-known as *Neumann* ($J = -1$) and *Robin boundary conditions* ($J = 0$) and for $J = 1$ possibly less known as *Wentzell boundary conditions* (see [6] and the references therein). The discretization of second order PDEs with local ABCs by the usual C^0 -continuous finite element methods (FEM) has been so far restricted to these three cases (see, e.g., [7] for Wentzell's conditions). For local ABCs of any order J finite element methods with $C^{(J-1)}$ -continuous basis functions (at least) along Γ [8,4] or with auxiliary unknowns [9,10] for each $\partial_\Gamma^2 u$, $\partial_\Gamma^4 u$ up to $\partial_\Gamma^{2(J-1)} u$ leading to a mixed system have been proposed. Even so, local ABCs with derivatives higher than two have rarely been used so far.

In this article, C^0 -continuous finite element methods were proposed for local symmetric ABCs of any order J , for both $d = 2, 3$, and are analysed when they are applied to the Helmholtz equation. These finite element methods are inspired by interior penalty discontinuous Galerkin methods [11] and exhibit additional terms on boundary nodes for $d = 2$ or boundary edges for $d = 3$. A similar approach was introduced by Brenner and Sung [12] for fourth order partial differential equations. Moreover, additional terms are introduced in the formulation if also tangential derivatives of odd orders are present, as this is the case for several impedance boundary conditions. Eventually, for $d = 3$ even more general boundary conditions with higher tangential derivatives applied to the Neumann trace are considered.

The article is organized as follows. In Section 2 several examples of local ABCs are given and corresponding variational formulations are introduced. Then, interior penalty formulations in two dimensions are introduced in Section 3 and in three dimensions in Section 4. The numerical analysis of the numerical method proposed in the article is presented for the case of local symmetric ABCs in two dimensions in Section 5. Finally, in Section 6 the proven theoretical convergence results are validated by a series of numerical experiments.

2. Local absorbing boundary conditions

Local ABCs are stated for example on artificial boundaries of truncated, originally infinite domains to approximate radiation or decay conditions [13,2]. Then, the functions α_j in (1) correspond to PDEs outside Ω and a better approximation is obtained by moving the artificial boundary further to infinity or by adding further terms, i.e., increasing J .

If a possibly bounded subdomain correspond to a highly conducting body in electromagnetics, the fields can be computed approximately by a formulation in the exterior of the conductor with so-called surface or generalized impedance boundary conditions [14,1,3] on the conductor surface. While introduced first by Rytov [15] and Leontovich [16], in recent years impedance boundary conditions of higher orders are proposed [17,18]. Similar impedance boundary conditions were derived for thin dielectric coatings on perfect conducting bodies [19–22] or for viscosity boundary layers in acoustics [23]. Furthermore, impedance transmission conditions may be used to approximate the behaviour of thin layers (see [24] and the references therein) or even microstructured layers [25]. In this case they relate jumps and means of Dirichlet and Neumann traces on the mid-surface of the layer.

The derivation of these local ABCs is often performed by asymptotic expansion techniques or by a truncation of Fourier series, where, at least for the rigorous error estimates, the boundary Γ and the functions α_j are assumed to be smooth. The local ABCs may also be applied for piecewise smooth boundaries Γ , e.g., domains with corners, or for piecewise smooth functions α_j which may have jumps. In this case the higher surface derivatives $\partial_\Gamma^j \alpha_j \partial_\Gamma^j$ are not necessary weak derivatives on the whole boundary Γ and additional conditions on the corners are needed [26]. To knowledge of the authors of this article, those corner conditions have not been mathematically analysed so far and this analysis is restricted to $\Gamma \in C^\infty$ with ring topology and $\alpha_j \in C^\infty$, $j = 0, \dots, J$.

With these assumptions, the weak form of (1), after j -time integration by parts of the j th term along Γ is, with v smooth enough, given by

$$\int_\Gamma \partial_\nu u v + \sum_{j=0}^J \alpha_j \partial_\Gamma^j u \partial_\Gamma^j v \, d\sigma(x) = \int_\Gamma g v \, d\sigma(x). \quad (2)$$

If the local ABCs are used in combination with the Helmholtz equation with homogeneous Neumann boundary conditions on $\partial\Omega \setminus \Gamma$ the corresponding variational formulation reads: Seek $u_j \in V_j := \{v \in H^1(\Omega) : v|_\Gamma \in H^j(\Gamma)\}$ such that

$$a_j(u_j, v) := \int_\Omega (\nabla u_j \cdot \nabla \bar{v} - \kappa^2 u_j \bar{v}) \, dx + \sum_{j=0}^J \int_\Gamma \alpha_j \partial_\Gamma^j u_j \partial_\Gamma^j \bar{v} \, d\sigma(x) = \langle f_j, v \rangle \quad \forall v \in V_j, \quad (3)$$

where f_j corresponds to source terms in the domain Ω or on the boundary Γ and the wave number $\kappa \in L^\infty(\Omega)$.

If only second derivatives are present in (1) and so only first derivatives in (3), i.e., for the Neumann, Robin and Wentzell conditions, a numerical realization with usual piecewise continuous finite element methods is straightforward. For $J \geq 2$, the usual finite element spaces are not contained any more in the natural space V_J of the continuous formulation. For those high-order conditions, finite elements methods with C^0 -continuous basis functions will be introduced in the following section.

3. Interior penalty finite element formulation in 2D

For the derivation of the interior penalty formulation, the following regularity result is needed.

Lemma 3.1. Let $u_j \in V_j$ be solution of (3) with $\inf_{x \in \Gamma} |\alpha_j| > 0$ and $\kappa \in C^\infty(\Omega_\Gamma)$ in some neighbourhood $\Omega_\Gamma \subset \Omega$ of Γ . Then, $u_j \in C^\infty(\Gamma)$.

Proof. The proof is a simple generalization of the proof of Lemma 2.8 in [5] from circular to C^∞ boundary Γ , from constants α_j to $\alpha_j \in C^\infty$, and from constants κ in some neighbourhood of Γ to C^∞ -functions in such a neighbourhood. \square

3.1. Definition of the C^0 -continuous finite element spaces

The presented non-conforming finite element method is based on a mesh \mathcal{M}_h of the computational domain Ω (see Fig. 1) consisting of possibly curved triangles \mathcal{T}_h and curved quadrilaterals \mathcal{Q}_h , which are disjoint and which fill the computational domain, i.e., $\Omega = \bigcup_{K \in \mathcal{M}_h} \bar{K}$. Each cell K in \mathcal{T}_h or \mathcal{Q}_h can be represented through a smooth mapping F_K from a single reference triangle \hat{K} or a single reference quadrilateral \hat{K} , respectively. The set of edges of \mathcal{M}_h on Γ is denoted by $\mathcal{E}(\mathcal{M}_h, \Gamma)$, $\mathcal{N}(\mathcal{M}_h, \Gamma)$ is the set of nodes of \mathcal{M}_h on Γ , and $\mathcal{N}(e)$ is the set composed of the two nodes of the external edge e . Furthermore, the union of all outer boundary edges and the union of all cells is defined as

$$\Gamma_h := \bigcup_{e \in \mathcal{E}(\mathcal{M}_h, \Gamma)} e = \Gamma \setminus \bigcup_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} n, \quad \Omega_h := \bigcup_{K \in \mathcal{M}_h} K = \Omega \setminus \bigcup_{e \in \mathcal{E}(\mathcal{M}_h)} \bar{e}.$$

The edges $\mathcal{E}(\mathcal{M}_h, \Gamma)$ of \mathcal{M}_h are possibly curved, where for the analysis they are assumed to resolve Γ exactly, i.e.,

$$\Gamma = \bigcup_{e \in \mathcal{E}(\mathcal{M}_h, \Gamma)} \bar{e}.$$

Furthermore, each edge is assumed to have counter-clockwise orientation and can be represented by a smooth mapping F_e from the reference interval $(0, 1)$. The mesh width h is the largest outer diameter of the cells

$$h := \max_{K \in \mathcal{M}_h} \text{diam}(K).$$

The discretization space will be defined in the following. First, \hat{K} denotes a reference quadrilateral or triangle, respectively, and \hat{e} denotes either one edge of \hat{K} or the reference interval. Furthermore, $\mathbb{P}_p(\hat{K})$ denotes the space of polynomials of maximal total degree p for the reference triangle \hat{K} and of maximal degree p in each coordinate direction for the reference quadrilateral \hat{K} . The space $\mathbb{P}_p(\hat{K})$ can be decomposed into interior bubbles, edges bubbles to one of the edges and the nodal functions. The space of interior bubbles for the reference triangle is $\mathbb{P}_p(\hat{K}, 0) := \{\hat{v} \in \mathbb{P}_p(\hat{K}) : \hat{v}|_{\partial \hat{K}} = 0\}$ and the one of the edge bubbles related to an edge \hat{e} in \hat{K} is given by $\mathbb{P}_p(\hat{K}, \hat{e}) := \{\hat{v} \in \mathbb{P}_p(\hat{K}) : \hat{v}|_{\partial \hat{K} \setminus \hat{e}} = 0, \hat{v} = \hat{\ell}(\hat{e}) \hat{v}|_{\hat{e}}\}$, where $\hat{\ell}(\hat{e}) \in \mathbb{P}_1(\hat{K})$ is the linear lifting function with $\hat{\ell}(\hat{e})|_{\hat{e}} = 1$.

Now, let p be a function assigning each cell $K \in \mathcal{M}_h$ and each edge $e \in \mathcal{E}(\mathcal{M}_h, \Gamma)$ a polynomial order $p(K)$ or $p(e)$, respectively, which are all positive integers and $p(e) \geq p(K)$ if $e \subset \bar{K}$. Let $p := \min_{K \in \mathcal{M}_h} p(K) \geq 1$ and $p_\Gamma := \min_{e \in \mathcal{E}(\mathcal{M}_h, \Gamma)} p(e) \geq p$. In order to define the local solution space, $\mathcal{M}_h(K)$ denotes the set of those neighbouring cells K' in $\mathcal{M}_h \setminus \{K\}$ which have a common edge with K and $\mathcal{E}(K, \Gamma)$ denotes the edges of K which are on the domain boundary Γ . The local function space in K is the space of polynomials with maximal degree $p(K)$ on the reference element (in each direction for a quadrilateral) and with possibly additional edge bubbles related to neighbouring elements and edge bubbles related to boundary edges,

$$\mathbb{P}_p(K) := \left\{ v_h \in C^\infty(K) : v_h|_K \circ F_K \in \mathbb{P}_{p(K)}(F_K^{-1}K) \oplus \bigoplus_{K' \in \mathcal{M}_h(K)} \mathbb{P}_{\max(p(K), p(K'))}(F_K^{-1}K, F_K^{-1}(\bar{K} \cap \bar{K}')) \right. \\ \left. \oplus \bigoplus_{e \in \mathcal{E}(K, \Gamma)} \mathbb{P}_{p(e)}(F_K^{-1}K, F_K^{-1}e) \right\}.$$

Note that the respective space on the reference element \hat{K} is a subset of $\mathbb{P}_{p^*(K)}(\hat{K})$ for some $p^*(K) \geq p(K)$. Here, $p^*(K)$ is the polynomial degree to represent the basis functions on K . It is larger than $p(K)$ if $p(K')$ is larger for one of the neighbouring elements $K' \in \mathcal{M}_h(K)$ or if an edge bubble function related to a higher polynomial order on a boundary edge has to be represented.

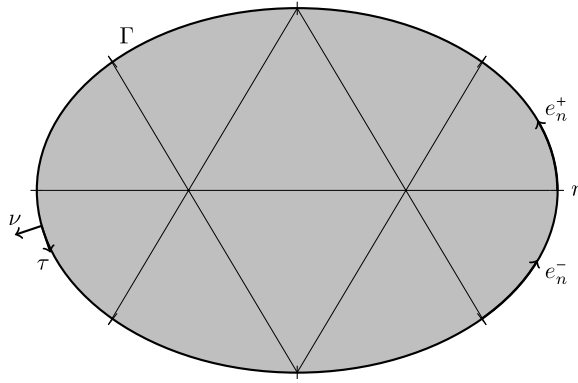


Fig. 1. The triangulation \mathcal{M}_h of the domain Ω with boundary Γ (unit normal vector ν and tangential vector τ are indicated) with partially curved cells. For functions on the boundary Γ the jump and average on boundary nodes n is defined using the neighbouring edges e_n^+ and e_n^- with counter-clockwise orientation.

Then, the space of continuous piecewise polynomial functions on \mathcal{M}_h with polynomial orders p is defined as

$$V_h := S^{p,1}(\Omega, \mathcal{M}_h) := \{v_h \in C^0(\overline{\Omega}) : v_h|_K \in \mathbb{P}_p(K)\}.$$

For the definition of the surface derivative, the tangential vector τ is assumed to be always the counter-clockwise (or clockwise) one on the whole Γ (see Fig. 1).

Finally, it is necessary to introduce additional notations specific to the non-conforming formulation. For each node $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$, e_n^+ and e_n^- denote the two external edges sharing n , such that e_n^+ follows e_n^- when going counter-clockwise (see Fig. 1). The trace of v on n taken from within e_n^+ and e_n^- is defined respectively by $v_n^+ = \lim_{s \rightarrow 0} v(F_{e_n^+}(s))$ and by $v_n^- = \lim_{s \rightarrow 1} v(F_{e_n^-}(s))$. The jump and the mean of v on n are respectively defined by

$$[v]_n = v_n^- - v_n^+ \quad \text{and} \quad \{v\}_n = \frac{v_n^+ + v_n^-}{2}.$$

The length of the edge e is denoted by h_e and h_n defines the smallest length of the two edges sharing the node n , i.e. $h_n = \min(h_{e_n^+}, h_{e_n^-})$. Furthermore, $h_\Gamma = \max_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} h_n$ denotes the mesh width of Γ_h and its quasi-uniformity measure is defined as $\mu_{\Gamma_h} = \max\{h_e/h_{e'}, e, e' \in \mathcal{E}(\mathcal{M}_h, \Gamma)\}$.

3.2. Derivation of the interior penalty Galerkin variational formulation

First, the term $\partial_\Gamma^j \alpha_j \partial_\Gamma^j u$ for a function $u \in C^\infty$ is multiplied by $\bar{v}_h \in V_h$, which is in $C^\infty(\bar{e})$ in each edge on Γ . Then, j times integration by part is successively applied in each edge e to obtain

$$\begin{aligned} (-1)^j \int_\Gamma \partial_\Gamma^j \alpha_j \partial_\Gamma^j u \bar{v}_h \, d\sigma(x) &= (-1)^j \sum_{e \in \mathcal{E}(\mathcal{M}_h, \Gamma)} \int_e \partial_\Gamma^j \alpha_j \partial_\Gamma^j u \bar{v}_h \, d\sigma(x) \\ &= \sum_{e \in \mathcal{E}(\mathcal{M}_h, \Gamma)} \int_e \alpha_j \partial_\Gamma^j u \partial_\Gamma^j \bar{v}_h \, d\sigma(x) + \sum_{i=0}^{j-1} (-1)^{i+j} \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} [\partial_\Gamma^{j-i-1} \alpha_j \partial_\Gamma^j u \partial_\Gamma^i \bar{v}_h]_n \\ &= \int_\Gamma \alpha_j \partial_\Gamma^j u \partial_\Gamma^j \bar{v}_h \, d\sigma(x) + \sum_{i=1}^{j-1} (-1)^{i+j} \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \{\partial_\Gamma^{j-i-1} \alpha_j \partial_\Gamma^j u\}_n [\partial_\Gamma^i \bar{v}_h]_n. \end{aligned} \quad (4)$$

Here, the equivalence $[ab]_n = [a]_n \{b\}_n + \{a\}_n [b]_n$ is exploited, the fact that with $u_j, \alpha_j \in C^\infty$ all jumps $[\partial_\Gamma^{j-i-1} \alpha_j \partial_\Gamma^j u]_n$, $i < j$ are zero and that with $v_h \in C^0(\Gamma)$ all jumps $[\bar{v}]_n$ are zero.

If one is interested in symmetric bilinear forms to obtain the symmetric interior penalty Galerkin formulation (SIPG) [27], it is necessary to add the terms $s [\partial_\Gamma^j u]_n \{\partial_\Gamma^{j-i-1} \alpha_j \partial_\Gamma^j \bar{v}_h\}_n$ with $s = 1$ leading to

$$\begin{aligned} (-1)^j \int_\Gamma \partial_\Gamma^j \alpha_j \partial_\Gamma^j u \bar{v}_h \, d\sigma(x) &= \int_\Gamma \alpha_j \partial_\Gamma^j u \partial_\Gamma^j \bar{v}_h \, d\sigma(x) \\ &+ \sum_{i=1}^{j-1} (-1)^{i+j} \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \left(\{\partial_\Gamma^{j-i-1} \alpha_j \partial_\Gamma^j u\}_n [\partial_\Gamma^i \bar{v}_h]_n + s [\partial_\Gamma^i u]_n \{\partial_\Gamma^{j-i-1} \alpha_j \partial_\Gamma^j \bar{v}_h\}_n \right). \end{aligned} \quad (5)$$

There is no loss of consistency as with the assumption of $u \in C^\infty(\Gamma)$ the terms $[\partial_r^j u]_n$ are in fact zero. Note that, alternatively, $s = -1$ for the non-symmetric (NIPG) [28] or $s = 0$ for the incomplete interior penalty Galerkin formulation (IIPG) [29] can be chosen.

Finally, to ensure the coercivity of the bilinear forms (with a mesh-independent constant), it remains to add for $j > 0$ the terms

$$\frac{\beta_j}{h_n^{2(j-j)+1}} [\partial_r^{j-1} u]_n [\partial_r^{j-1} \bar{v}_h]_n,$$

which also do not harm the consistency since $[\partial_r^{j-1} u]_n = 0$, $j = 1, \dots, J-1$.

Remark 3.2. The assumption $u \in C^\infty(\Gamma)$ in the derivation can be lowered. In fact $u \in H^1(\Gamma)$, $\alpha_0 u \in L^2(\Gamma)$, $\alpha_j \partial_r^j u \in C^{j-1}(\Gamma) \cap L^2(\Gamma)$, $j = 1, \dots, J$ is enough to ensure consistency. This requires $\alpha_j \in L^\infty(\Gamma)$, $j = 0, \dots, J$ and with $\partial_r^j u \in C^{j-1}(\Gamma)$ for $2j \leq J$ that $\alpha_j \in C^{j-1}(\Gamma)$, $j = 1, \dots, \lfloor \frac{J}{2} \rfloor$. With these assumptions it is indeed enough to require Γ to be Lipschitz and $C^{1,1}$ in a finite partition of the boundary. However, if u is solution of the above system it is unlikely to fulfil the regularity assumptions in this case [30].

Now, the interior penalty Galerkin formulation can be stated: Seek $u_{j,h} \in V_h$ such that

$$a_{j,h}(u_{j,h}, v_h) = (f_j, v_h), \quad \forall v_h \in V_h, \quad (6)$$

where

$$a_{j,h}(u_h, v_h) := \int_{\Omega} (\nabla u_h \cdot \nabla \bar{v} - \kappa^2 u_h \bar{v}_h) \, dx + \sum_{j=0}^J (c_j(u_h, v_h; \alpha_j) + b_{j,h;J}(u_h, v_h; \alpha_j))$$

$$c_j(u_h, v_h; \alpha_j) := \int_{\Gamma_h} \alpha_j \partial_r^j u_h \partial_r^j \bar{v}_h \, d\sigma(x),$$

$b_{0,h;J} = b_{1,h;J} = 0$ and for $j > 1$

$$\begin{aligned} b_{j,h;J}(u_h, v_h; \alpha_j) &:= \sum_{i=1}^{j-1} (-1)^{i+j} \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \left(\{ \partial_r^{j-i-1} \alpha_j \partial_r^j u_h \}_n [\partial_r^i \bar{v}_h]_n + s [\partial_r^i u_h]_n \{ \partial_r^{j-i-1} \alpha_j \partial_r^j \bar{v}_h \}_n \right) \\ &\quad + \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \frac{\beta_j}{h_n^{2(j-j)+1}} [\partial_r^{j-1} u_h]_n [\partial_r^{j-1} \bar{v}_h]_n, \end{aligned}$$

where s corresponds to the symmetric, non-symmetric or incomplete interior penalty method.

3.3. Well-posedness and estimation of the discretization error

If the finite element space is rich enough, the interior penalty formulation is well-posed and the discretization error can be bounded as stated in the following two theorems. For this the so called broken norm

$$\|v\|_{V_{j,h}}^2 := \|v\|_{H^1(\Omega)}^2 + \|v\|_{H^j(\Gamma_h)}^2 \quad (7)$$

is needed, as functions in V_h do not possess weak derivatives of order $j = 2, \dots, J$ in the whole Γ , but only in the open set Γ_h . Note, that for functions in V_j the $\|\cdot\|_{V_j}$ -norm and the $\|\cdot\|_{V_{j,h}}$ -norm coincide. The proofs of the following two theorems will be postponed to Section 5.

Theorem 3.3. Let $\inf_{x \in \Gamma} \operatorname{Re}(\alpha_j) > 0$ or $\inf_{x \in \Gamma} |\operatorname{Im}(\alpha_j)| > 0$ and let zero be the only solution of (3) with $f_j = 0$. Then, there exist constants $h_{\text{unique}}, p_{\text{unique}} > 0$ such that for all $h < h_{\text{unique}}$ and $p \geq p_{\text{unique}}$ the discrete interior penalty Galerkin variational formulation (6) for $s \in [-1, 1]$ and β_j , $j = 2, \dots, J$ large enough admits a unique solution $u_{j,h} \in V_h$ and there exists a constant $C_{j,h} > 0$ such that

$$\|u_{j,h}\|_{V_{j,h}} \leq C_{j,h} \|f_j\|_{V_j}. \quad (8)$$

Theorem 3.4. Let $J > 0$, let the assumption of Theorem 3.3 be satisfied, let $h < h_{\text{unique}}$, $p \geq p_{\text{unique}}$, $p_r \geq J$ and let the boundary mesh Γ_h be quasi-uniform, i.e., $\mu_{\Gamma_h} < \mu_{\Gamma}$ for some constant μ_{Γ} . Then, there exists a constant $C_j > 0$ independent of V_h such that for the solution $u_{j,h} \in V_h$ of (6) it holds

$$\|u_{j,h} - u_j\|_{V_{j,h}} \leq C_j \left(\inf_{v_h \in V_h} \|v_h - u_j\|_{H^1(\Omega)} + h_{\Gamma}^{p_{\Gamma}-J+1} \|f_j\|_{V_j} \right). \quad (9)$$

The first term on the right hand side of (9) is the H^1 -best-approximation error in the computational domain. It can be systematically decreased towards zero by mesh refinement or by increasing the polynomial degrees, possibly adaptively, especially in case of material corners (see e.g. [31] for p - and hp -finite element methods). The second term is due to the discretization of the surface differential operators in the symmetric local absorbing boundary conditions. In order to achieve a convergent discretization, the minimum p_r of the polynomial degrees on Γ_h has to be chosen to be at least J . For simple refinement of uniform meshes \mathcal{M}_h (h -refinement) and polynomial degrees of at least p in the cells of \mathcal{M}_h , the polynomial degrees on the edges of $\mathcal{E}(\mathcal{M}_h, \Gamma)$ has to be chosen to be at least $p_r \geq p + J - 1$ such that the error due to the discretization of the absorbing boundary condition does not dominate asymptotically for $h \rightarrow 0$.

3.4. Analysis of the computational costs

The proposed methodology requires $J - 1$ additional degrees of freedom per edge in $\mathcal{E}(\mathcal{M}_h, \Gamma)$ when comparing absorbing boundary conditions of order J to Neumann or Robin boundary conditions, while classical methodology requires the introduction of $J - 1$ auxiliary unknowns and thus of $p(J - 1)$ additional degrees of freedom per edge in $\mathcal{E}(\mathcal{M}_h, \Gamma)$. Note that, when considering odd order ABCs, the methodology proposed by Hagstrom et al. [32,33] reduces this cost to $(J - 1)/2$ auxiliary unknowns and $p(J - 1)/2$ degrees of freedom. Hence, the strategy developed in this paper is less costly and the higher the polynomial the larger the computational costs are reduced.

3.5. Interior penalty formulation for terms with odd tangential derivatives

The proposed interior penalty formulation can be extended to the local boundary condition involving terms with odd tangential derivatives of order $2J - 1$ and less. To illustrate this point, it is sufficient to derive the additional term in the variational formulation on the example of the term $\partial_r^2(\gamma \partial_r u)$ for $\gamma \in C^\infty(\Gamma)$. In analogy to terms with even derivatives, an integration by parts on Γ_h and the use of the fact that $[u]_n = 0$ on all $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$ leads to

$$\int_{\Gamma_h} \partial_r^2(\gamma \partial_r u) \bar{v} \, d\sigma(x) = - \int_{\Gamma_h} \partial_r(\gamma \partial_r u) \partial_r \bar{v} \, d\sigma(x).$$

Then, dividing the expression into two parts, applying integration by parts on one part, and using that $[\partial_r u]_n = 0$ on all $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$, it reads

$$\int_{\Gamma_h} \partial_r^2(\gamma \partial_r u) \bar{v} \, d\sigma(x) = \frac{1}{2} \int_{\Gamma_h} -\partial_r(\gamma \partial_r u) \partial_r \bar{v} + \gamma \partial_r u \partial_r^2 \bar{v} \, d\sigma(x) + \frac{1}{2} \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \gamma_n \{ \partial_r u \}_n [\partial_r \bar{v}]_n,$$

where γ_n are the function values of γ on $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$. Now, adding the terms $s\gamma_n \{ \partial_r u \}_n \{ \partial_r \bar{v} \}_n$ related to the different variants of interior penalty formulations and using the identity $\gamma \partial_r u \partial_r^2 \bar{v} = \partial_r u \partial_r(\gamma \partial_r \bar{v}) - \partial_r \gamma \partial_r u \partial_r \bar{v}$ it follows that

$$\begin{aligned} \int_{\Gamma_h} \partial_r^2(\gamma \partial_r u) \bar{v} \, d\sigma(x) &= \frac{1}{2} \int_{\Gamma_h} \partial_r u \partial_r(\gamma \partial_r \bar{v}) - \partial_r(\gamma \partial_r u) \partial_r \bar{v} \, d\sigma(x) - \frac{1}{2} \int_{\Gamma_h} (\partial_r \gamma) \partial_r u \partial_r \bar{v} \, d\sigma(x) \\ &\quad + \frac{1}{2} \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \gamma_n (\{ \partial_r u \}_n [\partial_r \bar{v}]_n + s [\partial_r u]_n \{ \partial_r \bar{v} \}_n). \end{aligned}$$

Obviously, the formulation with those additional terms related to odd derivatives loses symmetry even for $s = 1$. As only the highest derivatives in the formulation are crucial for coercivity no need to add any further penalty term to ensure coercivity, whatever s is chosen.

4. Interior penalty finite element formulation in 3D

4.1. Local absorbing boundary conditions in 3D

In three dimensions, the gradient of a function u can be decomposed into a contribution normal to the smooth surface Γ , that is $v \partial_v u = v(\nabla u \cdot v)$, and a tangential gradient $\nabla_r u := \nabla u - v \partial_v u$. Similarly, the Laplacian Δu can be decomposed into the second normal derivative $\partial_v^2 u = v^\top H(u) v$, where H is the Hessian matrix with all partial second derivatives, and the Laplace–Beltrami operator $\Delta_r u := \Delta u - \partial_v^2 u$. Note, that the latter is also given in terms of the surfacic divergence div_r by $\Delta_r := \text{div}_r \nabla_r$ (see [34, Sect. 2.5.6] for their definition on smooth surfaces).

For example, the ABCs by Bayliss, Gunzburger and Turkel [35] (BGT) set on a spherical boundary Γ of radius R are written in terms of the Laplace–Beltrami operator. For given wave number k , the BGT conditions of order 2 amount to

$$\partial_v u = \frac{1}{2} \left(-ik + \frac{2}{R} \right)^{-1} \left(\Delta_r + \frac{2}{R^2} + \frac{4ik}{R} + 2k^2 \right) u =: \mathcal{B}_2 u, \quad (10)$$

Table 1

Coefficients of the BGT conditions.

Order	α_0	α_1	α_2	β_0	β_1
\mathcal{B}_1	$-(-ik + \frac{1}{R})$	–	–	1	–
\mathcal{B}_2	$2(-ik + \frac{1}{R})^2$	1	–	$-2(-ik + \frac{2}{R})$	–
\mathcal{B}_3	$2(-2ik^3 + \frac{9k^2}{R} + \frac{9ik}{R^2} - \frac{3}{R^3})$	$-3(-ik + \frac{1}{R})$	–	$4(-ik + \frac{3}{R})(-ik + \frac{3}{2R})$	1
\mathcal{B}_4	$8(k^4 + \frac{8ik^3}{R} - \frac{18k^2}{R^2} - \frac{12ik}{R^3} + \frac{3}{R^4})$	$8(-ik + \frac{3}{R})^2$	1	$-8(-ik + \frac{2}{R})(-k^2 - \frac{6ik}{R} + \frac{6}{R^2})$	$-4(-ik + \frac{1}{R})$

where they serve as non-reflecting boundary conditions for the time-harmonic Helmholtz equation in 3D. If the shape of the boundary is arbitrary, but smooth, similar conditions were derived in [36]. These conditions include a term of the form $\text{div}_\Gamma(I - \frac{1}{k}\mathcal{R})\nabla_\Gamma$, where I is the identity and \mathcal{R} the curvature tensor. Non-reflecting boundary conditions for ellipsoidal boundaries were proposed in [37], and generalized impedance boundary conditions for highly conducting bodies of order 3 can be found in [17] in the form of Wentzell's conditions.

All these conditions can be discretized directly with C^0 -continuous finite elements with an additional bilinear form

$$\int_\Gamma \alpha uv + (\beta \nabla_\Gamma u) \cdot \nabla_\Gamma v dS(x),$$

only, where some scalar function α and some possibly tensorial function β appear. Patlashenko and Givoli [38,4] introduce symmetric local absorbing boundary conditions of any order J in three dimensions

$$\partial_\nu u = \sum_{j=0}^J \alpha_j (-1)^j \Delta_\Gamma^j u, \quad (11)$$

where α_j are scalar constants. In this case, (11) can be seen as a generalization of (1) with constant parameters α_j in three dimensions. The parameters α_j were computed by Harari in [39] in the specific case where the artificial boundary is a sphere.

To the best knowledge of the authors of this article, the usage of local absorbing boundary conditions with higher tangential derivatives than two in a finite element context has only been reported with basis functions with higher regularity on Γ , but not with the usual C^0 -continuous basis functions only.

The following is devoted to a more general case, where the Laplace–Beltrami operator is applied once or more to the normal derivative $\partial_\nu u$, and the highest derivative of u and $\partial_n u$ have same order. The conditions have then the form

$$0 = \sum_{j=0}^J (-1)^j \left(\alpha_j \Delta_\Gamma^j u + \beta_j \Delta_\Gamma^j \partial_\nu u \right), \quad (12)$$

where $\beta_0 \neq 0$. Hence, it can be assumed without loss of generality that $\beta_0 = -1$. Those conditions arise in the derivation of robust impedance conditions from a Padé approximation [17]. However, also the BGT conditions [35] of odd orders can be written under this form, as illustrated in Table 1. Note that the BGT conditions of order 1 and 2 can be written as (11).

In order to derive a general weak formulation for (11) or (12), the vector valued function $\Delta_\Gamma^{1/2} u := \nabla_\Gamma u$ is introduced, by misuse of notation. In addition, the vector-valued function $\Delta_\Gamma^{j/2} u$ for odd integer j is defined as $\Delta_\Gamma^{j/2} u := \nabla_\Gamma \Delta_\Gamma^{j-1/2} u$. Then, using Green formulae, the following identity holds for any $j \in \mathbb{N}$ and for functions $u, v \in C^\infty(\Gamma)$:

$$\int_\Gamma (-1)^j \Delta_\Gamma^j uv dS(x) = \int_\Gamma \Delta_\Gamma^{j/2} u \cdot \Delta_\Gamma^{j/2} v dS(x),$$

where \cdot denotes the dot product if j is odd and the usual product if j is even. With the Hilbert spaces in three dimensions $H_{\Delta_\Gamma}^j(\Gamma) := \{\Delta_\Gamma^{j/2} v \in L^2(\Gamma), j = 0, \dots, J\}$ and $V_j := \{v \in H^1(\Omega) : v|_\Gamma \in H_{\Delta_\Gamma}^j(\Gamma)\}$ the weak formulation for the Helmholtz equation with local ABCs (11) reads: Seek $u \in V_j$ such that

$$\int_\Omega (\nabla u \cdot \nabla \bar{v} - \kappa^2 u \bar{v}) dx + \sum_{j=0}^J \alpha_j \int_\Gamma \Delta_\Gamma^{j/2} u \cdot \Delta_\Gamma^{j/2} \bar{v} dS(x) = \langle f_j, v \rangle \quad \forall v \in V_j, \quad (13)$$

where f_j corresponds to the source terms. If $\text{Re}(a_j) > 0$ or $|\text{Im}(a_j)| > 0$, then the bilinear form a_j can be written as the sum of a V_j -elliptic bilinear form $a_{j,0}$ and a bilinear form k with only lower derivatives corresponding to a compact operator in V_j . Hence, there exists a Gårding inequality and, applying the Fredholm alternative, uniqueness of (13) implies existence of a solution (similar to [5, Chap. 2]).

The usual way to incorporate conditions with derivatives on the normal trace is the derivation of mixed formulations which introduce a new unknown $\lambda := \partial_\nu u$ and take the condition in weak form as an additional equation. In this work the

condition (12) is incorporated directly to the original equation. Then, defining the two bilinear forms

$$\begin{aligned} b_{J,0}((u, \lambda), v) &:= \int_{\Omega} (\nabla u \cdot \nabla \bar{v} - \kappa^2 u \bar{v}) \, dx + \sum_{j=0}^J \alpha_j \int_{\Gamma} \Delta_{\Gamma}^{j/2} u \cdot \Delta_{\Gamma}^{j/2} \bar{v} \, dS(x) + \sum_{j=1}^J \beta_j \int_{\Gamma} \Delta_{\Gamma}^{j/2} \lambda \cdot \Delta_{\Gamma}^{j/2} \bar{v} \, dS(x), \\ b_{J,1}((u, \lambda), \lambda') &:= \int_{\Gamma} \left(\lambda \bar{\lambda}' - \sum_{j=0}^J \alpha_j \Delta_{\Gamma}^{j/2} u \cdot \Delta_{\Gamma}^{j/2} \bar{\lambda}' - \sum_{j=1}^J \beta_j \Delta_{\Gamma}^{j/2} \lambda \cdot \Delta_{\Gamma}^{j/2} \bar{\lambda}' \right) dS(x), \end{aligned}$$

the mixed variational formulation for the Helmholtz equation with (12) on Γ reads as: Seek $(u, \lambda) \in V_J \times H_{\Delta\Gamma}^J(\Gamma)$ such that

$$b_J((u, \lambda), (v, \lambda')) := b_{J,0}((u, \lambda), v) + b_{J,1}((u, \lambda), \lambda') = \langle f_J, v \rangle \quad \forall (v, \lambda') \in V_J \times H_{\Delta\Gamma}^J(\Gamma). \quad (14)$$

Using test functions $(v, 0)$ and $(0, \lambda')$ it is easy to see that the two equations in the volume and on the artificial boundary are separately enforced. Equivalently to the two separate equations, the variational formulation $b_{J,0}((u, \lambda), v) + \gamma \overline{b_{J,1}((u, \lambda), \lambda')} = \langle f_J, v \rangle$ can be considered, where the conjugate complex of the second equation is taken and $\gamma \neq 0$ is an arbitrary complex factor. Then, with the choice $v = u$, $\lambda' = \lambda$ and $\gamma = \frac{\beta_J}{\alpha_J}$, the mixed terms of highest order cancel out and

$$\begin{aligned} b_{J,0}((u, \lambda), u) + \frac{\beta_J}{\alpha_J} \overline{b_{J,1}((u, \lambda), \lambda)} &= |u|_{H^1(\Omega)}^2 + \alpha_J |u|_{H_{\Delta\Gamma}^J(\Gamma)}^2 - \frac{|\beta_J|^2}{\alpha_J} |\lambda|_{H_{\Delta\Gamma}^J(\Gamma)}^2 \\ &\quad + \langle \kappa u, u \rangle_{L^2(\Omega)} + \sum_{j=0}^{J-1} \alpha_j |u|_{H_{\Delta\Gamma}^j(\Gamma)}^2 + \frac{\beta_J}{\alpha_J} \left(\|\lambda\|_{L^2(\Gamma)}^2 - \sum_{j=1}^{J-1} \bar{\beta}_j |\lambda|_{H_{\Delta\Gamma}^j(\Gamma)}^2 \right) \\ &\quad + \sum_{j=1}^{J-1} \left(\beta_j - \frac{\beta_J}{\alpha_J} \bar{\alpha}_j \right) \langle \Delta_{\Gamma}^{j/2} \lambda, \Delta_{\Gamma}^{j/2} u \rangle_{L^2(\Gamma)} - \frac{\beta_J}{\alpha_J} \bar{\alpha}_0 \langle \lambda, u \rangle. \end{aligned}$$

Here, the seminorms $|\cdot|_{H_{\Delta\Gamma}^j(\Gamma)} := \|\Delta_{\Gamma}^{j/2} \cdot\|_{L^2(\Gamma)}$, $j = 0, \dots, J$ were used. If $|\operatorname{Im}(\alpha_j)| > 0$ and $\beta_j \neq 0$ then there holds a Gårding inequality, i.e., there exists a constant $\theta \in (-\pi, \pi)$ and

$$\operatorname{Re} \left(e^{i\theta} \left(b_{J,0}((u, \lambda), u) + \frac{\beta_J}{\alpha_J} \overline{b_{J,1}((u, \lambda), \lambda)} \right) \right) \geq \gamma \left(\|u\|_{V_J}^2 + \|\lambda\|_{H_{\Delta\Gamma}^J(\Gamma)}^2 \right) - \delta \left(\|u\|_{W_{J-1}}^2 + \|\lambda\|_{H_{\Delta\Gamma}^{J-1}(\Gamma)}^2 \right),$$

for some constants $\gamma > 0$ and $\delta \in \mathbb{R}$, where $W_{J-1} := L^2(\Omega) \cap H_{\Delta\Gamma}^{J-1}(\Gamma)$. Since $V_J \subset \subset W_{J-1}$ and $H_{\Delta\Gamma}^J(\Gamma) \subset \subset H_{\Delta\Gamma}^{J-1}(\Gamma)$ the Fredholm alternative applies as well and there exists a unique solution of (14), except for a set of spurious eigenmodes.

4.2. Definition of the C^0 -continuous finite element spaces

Similarly to the two-dimensional case, the non-conforming finite element method is based on a mesh \mathcal{M}_h of the computational domain Ω consisting of possibly curved tetrahedra, hexahedra, prism or pyramids, which are disjoint and which fill the computational domain, i.e., $\Omega = \bigcup_{K \in \mathcal{M}_h} K$. The set of faces (triangles or quadrilaterals) of \mathcal{M}_h on Γ is denoted by $\mathcal{F}(\mathcal{M}_h, \Gamma)$, $\mathcal{E}(\mathcal{M}_h, \Gamma)$ is the set of edges of \mathcal{M}_h on Γ and $\mathcal{E}(e)$ is the set composed of all the edges of the external boundary Γ . Furthermore, the union of all outer boundary faces and the union of all cells are defined as

$$\Gamma_h := \bigcup_{f \in \mathcal{F}(\mathcal{M}_h, \Gamma)} f = \Gamma \setminus \bigcup_{e \in \mathcal{E}(\mathcal{M}_h, \Gamma)} e, \quad \Omega_h := \bigcup_{K \in \mathcal{M}_h} K = \Omega \setminus \bigcup_{f \in \mathcal{F}(\mathcal{M}_h)} \bar{f}.$$

It is assumed that each face can be represented by a smooth mapping F_f from the reference triangle or quadrilateral \hat{F} . As in 2D, the mesh width h is the largest outer diameter of the cells and $V_h := S^{p,1}(\Omega, \mathcal{M}_h)$ denotes the space of piecewise continuous polynomial functions on \mathcal{M}_h with polynomial orders p .

Finally, additional notations specific to the non-conforming formulation are introduced. For each edge $e \in \mathcal{E}(\mathcal{M}_h, \Gamma)$, the two external faces sharing e are arbitrarily denoted by f_e^+ and f_e^- and the trace of v on e taken from within f_e^+ and f_e^- is defined respectively by v_e^+ and by v_e^- . Furthermore, on an edge e of a face $f \in \mathcal{F}(\mathcal{M}_h, \Gamma)$, the outward unit normal vector of f on e (orthogonal to v) is denoted by τ_e . The jump and the mean of a scalar function v on e are respectively defined by

$$[v]_e = v_e^- - v_e^+ \quad \text{and} \quad \{v\}_e = \frac{v_e^+ + v_e^-}{2},$$

while the jump and the mean of a vectorial function v on e are respectively defined by

$$[v]_e = v_e^- \cdot \tau_e^- - v_e^+ \cdot \tau_e^+ \quad \text{and} \quad \{v\}_e = \frac{v_e^- \cdot \tau_e^- + v_e^+ \cdot \tau_e^+}{2}.$$

The diameter of the face f is denoted by h_f and h_e defines the smallest diameter of the two faces sharing the edge e , i.e., $h_e = \min(h_{f_e^+}, h_{f_e^-})$.

4.3. Definition of the interior penalty Galerkin variational formulation

The construction of the interior penalty Galerkin formulation is similar to the 2D case. First, $\Delta_r^{j/2} \alpha_j \Delta_r^{j/2} u$ for a function $u \in C^\infty$ is multiplied by $\bar{v}_h \in V_h$, which is in $C^\infty(\bar{f})$ in each face f on Γ , and j -times integration by part are successively applied in each face f . Second, the equivalence $[ab]_e = [a]_e \{b\}_e + \{a\}_e [b]_e$ is exploited, the fact that with $u_j, \alpha_j \in C^\infty$ all jumps $[\Delta_r^{(j-i-1)/2} \alpha_j \Delta_r^{j/2} u]_e, i < j$ are zero and that with $v_h \in C^0(\Gamma)$ all jumps $[\bar{v}]_e$ are zero. Third, the terms $s \int_e [\Delta_r^{j/2} u]_e \{\Delta_r^{(j-i-1)/2} \alpha_j \Delta_r^{j/2} \bar{v}_h\}_e d\sigma(x)$ are added, with $s = 1$ for SIPG, $s = -1$ for NIPG and $s = 0$ for IPDG. There is no loss of consistency since the terms $[\Delta_r^{j/2} u]_e$ are in fact zero due to the assumption that $u \in C^\infty(\Gamma)$. Finally, to ensure the coercivity of the bilinear forms (with a mesh-independent constant), the terms

$$\frac{\beta_j}{h_e^{2(j-1)+1}} \int_e [\Delta_r^{j-1/2} u]_e [\Delta_r^{j-1/2} \bar{v}_h]_e d\sigma(x)$$

are added for $j > 0$, which does not harm the consistency since $[\Delta_r^{j-1/2} u]_e = 0, j = 1, \dots, J-1$.

The interior penalty Galerkin formulation reads then: Seek $u_{j,h} \in V_h$ such that

$$a_{j,h}(u_{j,h}, v_h) = (f_j, v_h), \quad \forall v_h \in V_h, \quad (15)$$

where

$$a_{j,h}(u_h, v_h) := \int_\Omega (\nabla u \cdot \nabla \bar{v} - \kappa^2 u \bar{v}) dx + \sum_{j=0}^J \left(\int_{\Gamma_h} \alpha_j \Delta_r^{j/2} u_h \cdot \Delta_r^{j/2} \bar{v}_h dS(x) + b_{j,h}(u_h, v_h) \right)$$

and $b_{0,h} = b_{1,h} = 0$ and for $j > 1$

$$\begin{aligned} b_{j,h}(u_h, v_h) := & \sum_{i=1}^{j-1} (-1)^{i+j} \sum_{e \in \mathcal{E}(\mathcal{M}_h, \Gamma)} \int_e \left(\{\Delta_r^{(j-i-1)/2} \alpha_j \Delta_r^{j/2} u_h\}_e [\Delta_r^{i/2} \bar{v}_h]_e + s [\Delta_r^{i/2} u_h]_e \{\Delta_r^{(j-i-1)/2} \alpha_j \Delta_r^{j/2} \bar{v}_h\}_e \right) d\sigma(x) \\ & + \sum_{e \in \mathcal{E}(\mathcal{M}_h, \Gamma)} \frac{\beta_j}{h_e^{2(j-1)+1}} \int_e [\Delta_r^{j-1/2} u_h]_e [\Delta_r^{j-1/2} \bar{v}_h]_e d\sigma(x). \end{aligned}$$

5. Analysis of the interior penalty formulation in 2D

5.1. Associated variational formulation for infinite-dimensional spaces

The objective of this subsection is, in analogy to [11], the definition of an interior-penalty Galerkin variational formulation which is identical to the discrete one for the discrete space V_h and which can be defined for infinite-dimensional function spaces as well. As the discrete space V_h is not contained in the continuous function spaces V_j for $j \geq 2$, it is necessary to use larger spaces $V_{j,h}$, which include both V_h and V_j . These infinite-dimensional spaces are defined by

$$V_{j,h} := \{v \in H^1(\Omega) : v|_\Gamma \in H^1(\Gamma) \cap H^j(\Gamma_h)\} \supset V_j. \quad (16)$$

Note, that the trace of functions $v \in V_{j,h}$ is continuous on Γ and their tangential derivatives of order 1 to $J-1$ are bounded, but may be discontinuous over the nodes $\mathcal{N}(\mathcal{M}_h, \Gamma)$. In addition to the broken norm defined in (7) the norm

$$\|v\|_{V_{j,h}, \text{IP}}^2 := \|v\|_{H^1(\Omega)}^2 + \|v\|_{H^j(\Gamma_h)}^2 + \sum_{j=2}^J \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \frac{1}{h_n^{2(j-1)+1}} |[\partial_r^{j-1} v]_n|^2 \quad (17)$$

with penalty terms on the boundary nodes n will be used. These norms are equivalent, however, with constants depending on the edge lengths h_n .

The discrete formulation (6) cannot directly be used with the function space $V_{j,h}$ since the terms $\{\partial_r^{j-i-1} \alpha_j \partial_r^j u\}_n$ are not well-defined for $2j-i > J$ if $u \in V_{j,h}$. In the discrete variational formulation, those terms occur only in product with the finitely many piecewise polynomials in V_h . An extension of these products to functions $u \in V_{j,h}$ can be defined using the lifting operators $\mathcal{L}_{j,i} : H^1(\Gamma_h) \rightarrow V_h, i, j \in \mathbb{N}$ by

$$\int_{\Gamma_h} \mathcal{L}_{j,i}(v) \alpha_j \partial_r^j w_h d\sigma(x) = \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} [v]_n \{\partial_r^{j-i-1} \alpha_j \partial_r^j w_h\}_n \quad \forall w_h \in V_h. \quad (18)$$

Then, any occurrence of $\sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \{\partial_r^{j-i-1} \alpha_j \partial_r^j u_h\}_n [\partial_r^i \bar{v}_h]_n$ or of its symmetric counterpart in the discrete formulation (6) is replaced by $\int_{\Gamma_h} \alpha_j \partial_r^j u \mathcal{L}_{j,i}(\partial_r^i \bar{v}) d\sigma(x)$ or by its symmetric counterpart, respectively.

Now, the interior penalty Galerkin variational formulation for the infinite-dimensional spaces $V_{j,h}$ reads as: Seek $\tilde{u}_j \in V_{j,h}$ such that

$$\tilde{a}_{j,h}(\tilde{u}_j, v) = \langle f_{j,h}, v \rangle, \quad \forall v \in V_{j,h}, \quad (19)$$

where

$$\tilde{a}_{j,h}(u, v) := \int_{\Omega} (\nabla u_j \cdot \nabla \bar{v} - \kappa^2 u_j \bar{v}) \, dx + \sum_{j=0}^J (c_j(u_h, v_h; \alpha_j) + \tilde{b}_{j,h,j}(u, v; \alpha_j))$$

and for $\tilde{b}_{0,h,j} = \tilde{b}_{1,h,j} = 0$ and for $j > 1$

$$\tilde{b}_{j,h,j}(u, v; \alpha_j) := \sum_{i=1}^{j-1} (-1)^{i+j} \int_{\Gamma} \alpha_j \mathcal{L}_{j,i}(\partial_{\Gamma}^i u) \partial_{\Gamma}^j \bar{v} + s \alpha_j \mathcal{L}_{j,i}(\partial_{\Gamma}^i \bar{v}) \partial_{\Gamma}^j u \, d\sigma(x) + \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \frac{\beta_j}{h_n^{2(j-j)+1}} [\partial_{\Gamma}^{j-1} u]_n [\partial_{\Gamma}^{j-1} \bar{v}]_n.$$

Note that, due to the definition of the lifting operators, $\tilde{b}_{j,h,j} = b_{j,h,j}$ on $V_h \times V_h$ and $\tilde{b}_{j,h,j} = 0$ on $V_j \times V_j$, as all jump terms and so all lifting operators vanish and the weak derivatives exist on the whole Γ , not only on Γ_h . Hence, $\tilde{a}_{j,h} = a_{j,h}$ on $V_h \times V_h$ and $\tilde{a}_{j,h} = a_j$ on $V_j \times V_j$.

5.2. Analysis of the associated variational formulation

The following lemma states the equivalence of (3) and (19).

Lemma 5.1. Let $\langle f_j, v \rangle = \int_{\Omega} f v \, dx + \int_{\Gamma} g v \, d\sigma(x)$ with $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$. Then, the formulations (3) and (19) possess the same solutions, i.e., if $u_j \in V_j$ is solution of (3), then it solves (19), and if $\tilde{u}_j \in V_{j,h}$ is solution of (19), then it solves (3).

Proof. If $J = 0, 1$, then the formulations (3) and (19) are identical, and the proof continues with $J \geq 2$. The proof is in two steps. First, it is proven that the solution $u_j \in V_j$ of (3) solves (19), and then, that the solution $\tilde{u}_j \in V_{j,h}$ of (19) solves (3).

(i) Let $u_j \in V_j$ be solution of (3). Choosing test functions $v \in C_c^\infty(\Omega) \subset H_0^1(\Omega)$ vanishing on $\partial\Omega$ in (3) and using the definition of weak derivatives, it follows that u_j solves

$$-\Delta u_j - \kappa^2 u_j = f \quad \text{in } \Omega. \quad (20a)$$

If $\partial\Omega \setminus \Gamma$ is non empty, test functions v can be chosen in $v \in H^1(\Omega)$ with $v \equiv 0$ on Γ . Then, using integration by parts in Ω and the fact that u_j solves (20a) one can show that u_j solves

$$\partial_\nu u_j = 0 \quad \text{on } \partial\Omega \setminus \Gamma. \quad (20b)$$

Now, taking test functions v in the whole space V_j , using integration by parts in Ω and on Γ , and using the fact that u_j solves (20a) and (20b) it holds in the same way

$$\partial_\nu u_j + \sum_{j=0}^J (-1)^j \partial_{\Gamma}^j (\alpha_j \partial_{\Gamma}^j u_j) = g \quad \text{on } \Gamma. \quad (20c)$$

Following the same steps as for the construction of the bilinear form $a_{j,h}$ (but using the lifting operators instead of the jump terms), it follows easily that u_j solves $\tilde{a}_{j,h}(u_j, v) = \langle f_{j,h}, v \rangle$ for all $v \in V_{j,h}$.

(ii) Let $\tilde{u}_j \in V_{j,h}$ be solution of (19). In the same way as in Part (i), it can be proven that \tilde{u}_j solves (20a) and (20b). Now, let the test function $v \in V_j \cap C_c^\infty(\Gamma_h)$ such that $b_j(u, v) = \tilde{b}_{j,h}(u, v)$ holds for any $u \in V_j$. Then using integration by parts in Ω , and the fact that \tilde{u}_j solves (20a) and (20b), it follows that \tilde{u}_j solves (20c) on Γ_h . Comparing with the integration by parts formula in (4) this proves that for all $v \in V_j \cap C^\infty(\Gamma_h)$

$$\begin{aligned} & \sum_{j=2}^J \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \left(\sum_{i=1}^{j-1} (-1)^{i+j} \left([\partial_{\Gamma}^{j-i-1} \alpha_j \partial_{\Gamma}^j \tilde{u}_j]_n \{ \partial_{\Gamma}^i \bar{v} \}_n - s [\partial_{\Gamma}^i \tilde{u}_j]_n \{ \partial_{\Gamma}^{j-i-1} \alpha_j \partial_{\Gamma}^j \bar{v} \}_n \right) \right. \\ & \left. + \frac{\beta_j}{h_n^{2(j-j)+1}} [\partial_{\Gamma}^{j-1} \tilde{u}_j]_n [\partial_{\Gamma}^{j-1} \bar{v}]_n \right) = 0. \end{aligned}$$

Let now the test function $v \in V_j \cap C^\infty(\Gamma_h)$ such that it holds $\{ \partial_{\Gamma}^{j-1} \bar{v} \}_n = 0$ for all $j = 2, \dots, J$ and all $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$ and such that $\{ \partial_{\Gamma}^{j-i-1} \alpha_j \partial_{\Gamma}^j \bar{v} \}_n = 0$ for all $j = 2, \dots, J$ and $i = 1, \dots, j-1$ if $s \neq 0$. Then it holds that

$$\sum_{j=2}^J \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \frac{\beta_j}{h_n^{2(j-j)+1}} [\partial_{\Gamma}^{j-1} \tilde{u}_j]_n [\partial_{\Gamma}^{j-1} \bar{v}]_n = 0.$$

This is only possible if $[\partial_{\Gamma}^{j-1} \tilde{u}_j]_n = 0$ for $j = 2, \dots, J$ and any $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$.

Hence, this proves that \tilde{u}_j is in V_j and solves (20) and so (3).

This completes the proof. \square

The following lemmata are required to prove the well-posedness of the variational formulation (19).

Lemma 5.2. *The lifting operators $\mathcal{L}_{j,i} : H^1(\Gamma_h) \rightarrow V_h$, $i, j \in \mathbb{N}$ defined by (18) are continuous, i.e., there exist constants $C_{j,i} > 0$ such that*

$$\|\mathcal{L}_{j,i}(v)\|_{L^2(\Gamma)}^2 \leq C_{j,i}^2 \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \frac{|[v]_n|^2}{h_n^{2(j-i)-1}}.$$

Proof. From the definition of the mean over n , it holds

$$\left| \left\{ \partial_r^{j-i-1} \alpha_j \partial_r^j w_h \right\}_n \right|^2 \leq \frac{\left| (\partial_r^{j-i-1} \alpha_j \partial_r^j w_h)_n^+ \right|^2 + \left| (\partial_r^{j-i-1} \alpha_j \partial_r^j w_h)_n^- \right|^2}{2}, \quad (21)$$

with the notation $(v)_n^\pm = v_n^\pm$. Now, using inverse inequalities [31,40] and the fact that $h_n \leq h_{e_n^\pm}$, it reads

$$\left| (\partial_r^{j-i-1} \alpha_j \partial_r^j w_h)_n^\pm \right|^2 \leq \frac{C_{j,i}^2}{2 h_n^{2(j-i)-1}} \int_{e_n^\pm} |\alpha_j \partial_r^j w_h|^2 d\sigma(x) \quad (22)$$

with $C_{j,i} := C_{j,i}(p_\Gamma) = O(p_\Gamma^{2(j-i)-1})$, and using (21) yields

$$\left| \left\{ \partial_r^{j-i-1} \alpha_j \partial_r^j w_h \right\}_n \right|^2 \leq \frac{C_{j,i}^2}{2 h_n^{2(j-i)-1}} \int_{e_n^+ \cup e_n^-} |\alpha_j \partial_r^j w_h|^2 d\sigma(x)$$

and so

$$\sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} h_n^{2(j-i)-1} \left| \left\{ \partial_r^{j-i-1} \alpha_j \partial_r^j w_h \right\}_n \right|^2 \leq C_{j,i}^2 \|\alpha_j \partial_r^j w_h\|_{L^2(\Gamma)}^2. \quad (23)$$

Then, using the definition of the lifting operator $\mathcal{L}_{j,i}$ and the Cauchy–Schwarz-inequality, it follows that

$$\begin{aligned} \|\mathcal{L}_{j,i}(v)\|_{L^2(\Gamma)}^2 &= \max_{w_h \in V_h} \frac{\left| \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} [v]_n \left\{ \partial_r^{j-i-1} \alpha_j \partial_r^j w_h \right\}_n \right|^2}{\|\alpha_j \partial_r^j w_h\|_{L^2(\Gamma)}^2} \\ &\leq \max_{w_h \in V_h} \frac{\sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \frac{|[v]_n|^2}{h_n^{2(j-i)-1}} \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} h_n^{2(j-i)-1} \left| \left\{ \partial_r^{j-i-1} \alpha_j \partial_r^j w_h \right\}_n \right|^2}{\|\alpha_j \partial_r^j w_h\|_{L^2(\Gamma)}^2}. \end{aligned}$$

Finally, inserting (23), and the statement of the lemma is proved. \square

Lemma 5.3. *Let $s \in [-1, 1]$, $J \geq 1$, $\alpha_j \in L^\infty(\Gamma)$ with $\inf_{x \in \Gamma} \operatorname{Re}(\alpha_j) > 0$ or $\inf_{x \in \Gamma} |\operatorname{Im}(\alpha_j)| > 0$. Then, for β_j large enough the bilinear form $\tilde{a}_{0,j}$ defined by*

$$\tilde{a}_{0,j}(u, v) := \int_{\Omega} (\nabla u \cdot \nabla \bar{v} + u \bar{v}) dx + \sum_{j=0}^{J-1} (c_{j,h}(u, v; 1) + \tilde{b}_{j,h;j}(u, v; 1)) + c_{j,h}(u, v; \alpha_j) + \tilde{b}_{j,h;j}(u, v; \alpha_j), \quad (24)$$

is $V_{j,h}$ -elliptic with an ellipticity constant independent of h_n for all $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$ when measured in the $\|\cdot\|_{V_{j,h}, \mathbb{P}}$ -norm.

The proof is a simple consequence of the following lemma, in which a more explicit statement how large the penalty terms β_j need to be is given.

Lemma 5.4. *Let the assumption on s and α_j in Lemma 5.3 be fulfilled. Then, there exist constants C , γ and θ , such that the following inequality holds for $\beta_j > \sqrt{j}C$, $j \geq 2$:*

$$\begin{aligned} &\operatorname{Re} \left(e^{i\theta} \sum_{j=0}^{J-1} (c_{j,h}(v, v; 1) + \tilde{b}_{j,h;j}(v, v; 1)) \right) + \operatorname{Re} \left(e^{i\theta} \left(c_{j,h}(v, v; \alpha_j) + \tilde{b}_{j,h;j}(v, v; \alpha_j) \right) \right) \\ &\geq \gamma \left(\|v\|_{H^j(\Gamma)}^2 + \sum_{j=2}^J \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \frac{1}{h_n^{2(j-1)+1}} \left| \left[\partial_r^{j-1} v \right]_n \right|^2 \right) \quad \forall v \in H^j(\Gamma_h). \end{aligned}$$

Proof. With the assumption on α_j there exists $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that it holds

$$\operatorname{Re} \left(e^{i\theta} \int_{\Gamma} \alpha_j |v|^2 \right) \geq \gamma_j |v|_{L^2(\Gamma)}^2, \quad (25)$$

with a positive constant γ_j . In the remainder of the proof it is assumed that θ is such that (25) is fulfilled. The first step of the proof is to write

$$\begin{aligned} \operatorname{Re} \left(e^{i\theta} \tilde{\mathbf{b}}_{j,h;J}(v, v; \alpha_j) \right) &= \sum_{i=1}^{j-1} (-1)^{i+j} \operatorname{Re} \left(e^{i\theta} \int_{\Gamma} \alpha_j \mathcal{L}_{j,i}(\partial_{\Gamma}^i v) \partial_{\Gamma}^j \bar{v} + s \alpha_j \mathcal{L}_{j,i}(\partial_{\Gamma}^i \bar{v}) \partial_{\Gamma}^j v \, d\sigma(x) \right) \\ &\quad + \cos(\theta) \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \frac{\beta_j}{h_n^{2(j-1)+1}} \left| \left[\partial_{\Gamma}^{j-1} v \right]_n \right|^2. \end{aligned}$$

Using that $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$ for any positive ε , the fact that $\alpha_j = 1$ for $j < J$ in the bilinear forms $\tilde{\mathbf{b}}_{j,h;J}$, the assumption on α_j and Lemma 5.2, the estimate

$$\begin{aligned} 2 \left| \int_{\Gamma} \alpha_j \mathcal{L}_{j,i}(\partial_{\Gamma}^i v) \partial_{\Gamma}^j \bar{v} \, d\sigma(x) \right| &\leq \varepsilon \|\partial_{\Gamma}^j v\|_{L^2(\Gamma)}^2 + \varepsilon^{-1} \|\alpha_j \mathcal{L}_{j,i}(\partial_{\Gamma}^i v)\|_{L^2(\Gamma)}^2 \\ &\leq \varepsilon \|\partial_{\Gamma}^j v\|_{L^2(\Gamma)}^2 + \varepsilon^{-1} \|\alpha_j\|_{L^\infty(\Gamma)}^2 C_{j,i}^2 \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \frac{1}{h_n^{2(j-1)+1}} \left| \left[\partial_{\Gamma}^i v \right]_n \right|^2, \end{aligned}$$

follows. Obviously, the same bound holds, if v and \bar{v} are interchanged. Consequently, it follows that

$$\begin{aligned} &\operatorname{Re} \left(e^{i\theta} \sum_{j=0}^{J-1} \left(c_{j,h}(v, v; 1) + \tilde{\mathbf{b}}_{j,h;J}(v, v; 1) \right) \right) + \operatorname{Re} \left(e^{i\theta} \left(c_{j,h}(v, v; \alpha_j) + \tilde{\mathbf{b}}_{j,h;J}(v, v; \alpha_j) \right) \right) \\ &\geq \cos(\theta) \|v\|_{H^1(\Gamma)}^2 + \sum_{j=2}^{J-1} (\cos(\theta) - (j-1)\varepsilon_j) \|\partial_{\Gamma}^j v\|_{L^2(\Gamma)}^2 + (\gamma_j - (J-1)\varepsilon_j) \|\partial_{\Gamma}^J v\|_{L^2(\Gamma)}^2 \\ &\quad + \sum_{j=2}^J \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \left(\frac{\cos(\theta)\beta_j}{h_n^{2(j-1)+1}} - \sum_{i=j}^J \varepsilon_i^{-1} \|\alpha_i\|_{L^\infty(\Gamma)} \frac{C_{i,j-1}^2}{h_n^{2(i-j)+1}} \right) \left| \left[\partial_{\Gamma}^{j-1} v \right]_n \right|^2. \end{aligned}$$

Finally, choosing $\varepsilon_j = \cos(\theta)/j$ for $j = 2, \dots, J-1$ and $\varepsilon_J = \gamma_J/J$, and with the assumption on β_j , the desired inequality is obtained. \square

The following two theorems state the main results for the variational formulation (19).

Theorem 5.5. Let the assumption of Lemma 5.4 be satisfied, and let zero be the only solution of (19) with $u^{\text{inc}} \equiv 0$. Then, there exists a unique solution $\tilde{u}_j \in V_{j,h}$ of (19) and there exists a constant $C_j > 0$ independent of h_n for all $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$ such that

$$\|\tilde{u}_j\|_{V_{j,h,\text{IP}}} \leq C_j \|f_{j,h}\|_{V'_{j,h,\text{IP}}}.$$

Proof. The proof is along the lines of that of [5, Theorem 2.3]. By Lemma 5.3 the bilinear form $\tilde{\mathbf{a}}_{0,J}$ is $V_{j,h}$ -elliptic and so the associated operators $\tilde{A}_{0,J}$ are isomorphisms in $V_{j,h}$. Let the Sobolev spaces

$$W_{0,h} := L^2(\Omega), \quad W_{j,h} := L^2(\Omega) \cap H^{j-1}(\Gamma_h), \quad j > 0,$$

be defined, where the Rellich–Kondrachov compactness theorem [41, Chap. 6] implies that the embedding $V_{j,h} \subset\subset W_{j,h}$ is compact. Now, the operators \tilde{K}_j associated to the bilinear forms

$$\tilde{k}_j(u, v) := - \int_{\Gamma} (\kappa^2 + 1) u \bar{v} \, dx + \sum_{j=0}^{J-1} (c_{j,h}(u, v; \alpha_j - 1) + \tilde{\mathbf{b}}_{j,h;J}(u, v; \alpha_j - 1)), \quad j > 0,$$

are compact.

Hence, the operators $\tilde{A}_{0,J} + \tilde{K}_j$ associated to the bilinear forms $\tilde{a}_j = \tilde{\mathbf{a}}_{0,J} + \tilde{k}_j$ are Fredholm with index 0 and by the Fredholm alternative [42, Sec. 2.1.4] the uniqueness of a solution of (3) implies its existence and its continuous dependence on the right hand side with constants independent of h_n for all $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$, and the proof is complete. \square

Theorem 5.6. Let the assumption of Theorem 5.5 be satisfied, and let $c^{-1} \in L_{\text{loc}}^\infty(\mathbb{R}^2)$ fixed with $c(x) = c_0 > 0$ for $|x| > R_C$ and $\kappa(x) = \omega/c(x)$ with the frequency $\omega > \mathbb{R}^+$. Then, (19) has a unique solution except for a countable (possibly finite) set of frequencies ω , the spurious eigenfrequencies, which accumulates only at infinity. The set of these frequencies coincides with the one of (3).

Proof. Using Lemma 5.1, the proof of this statement is analogue to the proof of [5, Lemma 2.6]. \square

Remark 5.7. As the solution $\tilde{u}_j \in V_{j,h}$ of (19) coincides with $u_j \in V_j$ of (3) and as the eigenfrequencies coincide, the guaranteed uniqueness in case of Feng's absorbing boundary conditions for large enough domains by [5, Lemma 2.7] apply to \tilde{u}_j as well.

5.3. Analysis of the discrete discontinuous Galerkin variational formulation

The discrete discontinuous Galerkin variational formulation (6) is the Galerkin discretization of the associated variational formulation (19), when using V_h as the finite-dimensional subspace of $V_{j,h}$.

Proof of Theorem 3.3. By the assumption that zero is the only solution of the continuous variational formulation (3) with zero sources, there exist a function $c(x)$ and a frequency ω such that $\kappa(x) = \omega/c(x)$ as in [5, Lemma 2.6] and such that ω is not a spurious eigenfrequency. For $J = 0$ and $|\operatorname{Im} \alpha_0| > 0$ the statement of the theorem was proved by Melenk and Sauter [43, Thm. 5.8].

The discrete system (6) is the Galerkin discretization of (19), which has by Theorem 5.6, with $\kappa(x) = \omega/c(x)$, the same eigenfrequencies as (3). Both systems are non-linear in ω and can be regarded in a similar fix-point form, as in [5, Eq. (2.6)]. These systems are linear eigenvalue problems in ω^2 for given parameter $\tilde{\omega} \in \mathbb{C} \setminus \{0\}$, where the fix-point system of (19) admits a countable set of frequencies $\omega_m(\tilde{\omega})$ and that of (6) a finite set $\omega_{m,h}(\tilde{\omega})$. As ω is not a spurious eigenfrequency, $\omega_m(\omega) \neq \omega$ for any $m \in \mathbb{N}$, and so the distances of the curves $\omega_m(\tilde{\omega})$ to the point $\tilde{\omega} = \omega$ are positive. Let $d_m, m \in \mathbb{N}$ denote these distances.

By the Babuška–Osborn theory [44] the discrete eigenfrequencies, $\omega_{m,h}(\tilde{\omega})$ tend to $\omega_m(\tilde{\omega})$ if the mesh-widths tend to zero for a minimal polynomial degree, which depends on J , or if the polynomial degrees tend to infinity. As a consequence, the distance $d_{m,h}$ of the curve $\omega_{m,h}(\tilde{\omega})$ to the point $\tilde{\omega} = \omega$ tends to d_m , and for a fine enough mesh or large enough polynomial degrees $|d_{m,h} - d_m| < \frac{1}{2}d_m$, and so $d_{m,h} > 0$. This means that ω is not an eigenfrequency of the discrete variational problem (6). Hence, it admits a unique solution [42, Sec. 2.1.6] bounded by (8).

This completes the proof. \square

Proof of Theorem 3.4. First, to follow up directly the proof of Theorem 3.3 the observation that the bilinear form $a_{j,h}$ of the interior penalty Galerkin formulation satisfies a Gårding inequality implies its asymptotic quasi-optimality in the $V_{j,h,\text{IP}}$ -norm, see [45,46] and [43, Sec. 3.2], i.e.,

$$\|u_{j,h} - u_j\|_{V_{j,h,\text{IP}}} \leq C \inf_{v_h \in V_h} \|v_h - u_j\|_{V_{j,h,\text{IP}}}, \quad (26)$$

where C denotes throughout the proof a constant not depending on V_h .

To estimate the jump terms in the definition of the $V_{j,h,\text{IP}}$ -norm, see (17), let on any edge $e \in \mathcal{E}(\mathcal{M}_h, \Gamma)$ the pullback of v_h and u_j to the reference interval $[0, 1]$ be denoted by $(\widehat{v}_h)_e$ and $(\widehat{u}_j)_e$. Then, using the trace theorem twice on the reference interval it follows for any $j = 2, \dots, J$ and any $n \in \mathcal{N}(\mathcal{M}_h, \Gamma)$ that

$$\begin{aligned} \frac{1}{h_n^{2(j-j)+1}} \left| \left[\partial_r^{j-1} (v_h - u_j) \right]_n \right|^2 &\leq \frac{C}{h_n^{2j-1}} \left(\left| \partial_s^{j-1} ((\widehat{v}_h)_{e_n^+} - (\widehat{u}_j)_{e_n^+})|_{s=0} \right|^2 + \left| \partial_s^{j-1} ((\widehat{v}_h)_{e_n^-} - (\widehat{u}_j)_{e_n^-})|_{s=1} \right|^2 \right) \\ &\leq \frac{C}{h_n^{2j-1}} \left(\|(\widehat{v}_h)_{e_n^+} - (\widehat{u}_j)_{e_n^+}\|_{H^j(0,1)}^2 + \|(\widehat{v}_h)_{e_n^-} - (\widehat{u}_j)_{e_n^-}\|_{H^j(0,1)}^2 \right) \\ &\leq C \sum_{\ell=0}^j \frac{1}{h_n^{2(j-\ell)}} \left(|v_h - u_j|_{H^\ell(e_n^+)}^2 + |v_h - u_j|_{H^\ell(e_n^-)}^2 \right), \end{aligned}$$

where s denotes the local coordinate in $[0, 1]$.

Hence, it follows for any $v_h \in V_h$ that

$$\begin{aligned} \|v_h - u_j\|_{V_{j,h,\text{IP}}}^2 &= \|v_h - u_j\|_{H^1(\Omega)}^2 + \sum_{j=1}^J |v_h - u_j|_{H^j(\Gamma_h)}^2 + \sum_{j=2}^J \sum_{n \in \mathcal{N}(\mathcal{M}_h, \Gamma)} \frac{1}{h_n^{2(j-j)+1}} \left| \left[\partial_r^{j-1} (v_h - u_j) \right]_n \right|^2 \\ &\leq C \left(\|v_h - u_j\|_{H^1(\Omega)}^2 + \sum_{j=0}^J \frac{1}{h_\Gamma^{2(j-j)}} |v_h - u_j|_{H^j(\Gamma_h)}^2 \right). \end{aligned} \quad (27)$$

To obtain optimal bounds for the terms on Γ_h the function v_h is replaced by the Raviart–Thomas interpolation operator $I_{\Gamma,h} u_j$ on the trace space $TV_h(\Gamma)$ of V_h on Γ , for which it holds [47, Chap. 3]

$$(h_\Gamma^{j-j})^{-1} |I_{\Gamma,h} u_j - u_j|_{H^j(\Gamma_h)} \leq C h_\Gamma^{p_\Gamma - j + 1} \|u_j\|_{H^{p_\Gamma+1}(\Gamma)} \leq C h_\Gamma^{p_\Gamma - j + 1} \|f_j\|_{V_j'}. \quad (28)$$

Here, [5, Lemma 2.8] was used to obtain the last inequality.

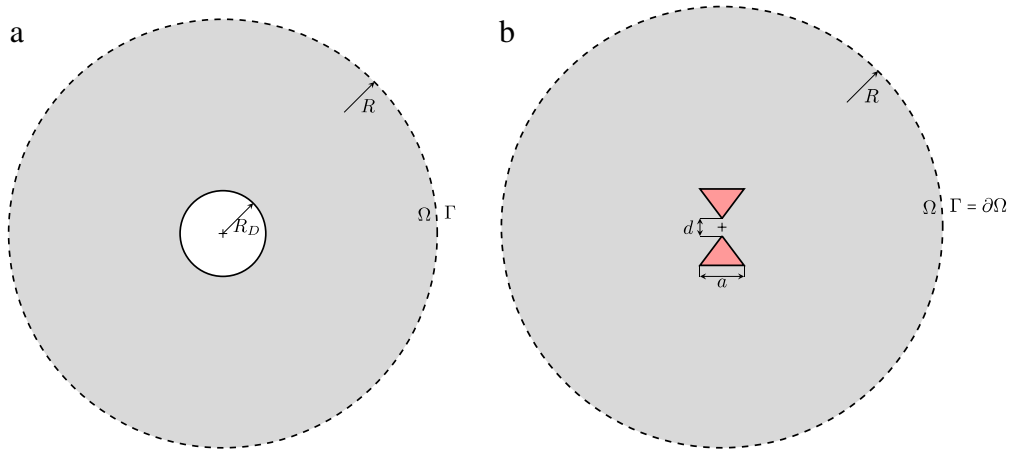


Fig. 2. The geometrical setting for the (a) acoustic scattering on a rigid cylinder of radius $R_D = 1$, and (b) for the electromagnetic scattering on two cylinders with equilateral triangles as cross-section of length $a = 1.05$ and distance $d = 0.25$. The outer boundary is a circle of radius R .

To estimate the first term in (27) the function v_h is replaced by the $H^1(\Omega)$ -projection $Q_{\Omega,h} : V_{J,h} \rightarrow V_h$, for which $Q_{\Omega,h} \cdot |_{\Gamma} = I_{\Gamma,h}$. Then,

$$\int_{\Omega} \nabla(Q_{\Omega,h} u_J - u_J) \cdot \nabla w_h + (Q_{\Omega,h} u_J - u_J) w_h \, dx = 0 \quad \forall w_h \in V_{h,0},$$

where $V_{h,0}$ is the subset of functions in V_h whose traces vanish on Γ . Since the projection $Q_{\Omega,h}$ is defined via the $H^1(\Omega)$ -inner product, its continuity follows from Lax–Milgram’s lemma

$$\|Q_{\Omega,h} u_J\|_{H^1(\Omega)} \leq \|u_J\|_{H^1(\Omega)}.$$

As by definition $Q_{\Omega,h}$ is a projection onto V_h it holds that

$$\begin{aligned} \|Q_{\Omega,h} u_J - u_J\|_{H^1(\Omega)} &\leq \inf_{w_h \in V_h} \|Q_{\Omega,h}(u_J - w_h) - (u_J - w_h)\|_{H^1(\Omega)} \\ &= \inf_{w_h \in V_h} \|(Q_{\Omega,h} - Id)(u_J - w_h)\|_{H^1(\Omega)} \leq 2 \inf_{w_h \in V_h} \|u_J - w_h\|_{H^1(\Omega)}. \end{aligned} \quad (29)$$

Finally, inserting $v_h = Q_{\Omega,h} u_J$ into (26) and using (27)–(29), and the proof is complete. \square

6. Numerical experiments

The non-conforming Galerkin formulation introduced in Section 3 for the Feng-4 and Feng-5 conditions was implemented in the numerical C++ library Concepts [48–50], as well as Feng-0 up to Feng-3 with the usual continuous formulations (compare [7] for implementational details related to BGT absorbing boundary conditions). The hp -FEM part of Concepts is based on quadrilateral, curved cells in 2D where the polynomial degree can be set independently in each cell and even anisotropically. The circular boundary can be exactly resolved with cells having circular edges. We approximate the integrals using numerical quadrature of sufficiently high order such that the contribution to the discretization error can be neglected.

For the numerical experiments, two model problems are studied:

- The acoustic scattering on a rigid cylinder with circular cross-section, where the computational domain Ω is the disc of radius R , with the values $R = 3$ or $R = 8$, without the disc of radius $R_D = 1$ (see Fig. 2(a)) and $k = 1$, and
- The electromagnetic scattering on two dielectric prisms, whose cross-section are equilateral triangles of length $a = 1.05$ and distance $d = 0.25$ (see [51] and Fig. 2(b)). The computational domain Ω is the disc of radius $R = 8$, and $\kappa^2(x) = \varepsilon(x)\omega^2$ with the angular frequency $\omega = 0.638$ and the (relative) dielectricity $\varepsilon(x)$, which is $-40.2741 - 2.794i$ inside the prisms and 1 outside. Hence $k = \omega$ and the wavelength $\lambda = 2\pi/k = 9.84$ in the exterior of the scatterer.

For both model problems the incident wave is a plane wave in direction $(1, 0)^T$ (from left). For model problem B the mesh is refined close to the nodes of the triangles. The scattered fields for the two problems are shown in Fig. 4.

Discretization error for model problem A. The discretization error for the Feng-5 condition for the model problem A is studied, where a family of meshes of the computational domain Ω with $R = 8$, shown in Fig. 3, are used for the computations. Reference solutions are computed for the same model with the Feng-5 condition and on the same mesh, respectively, but

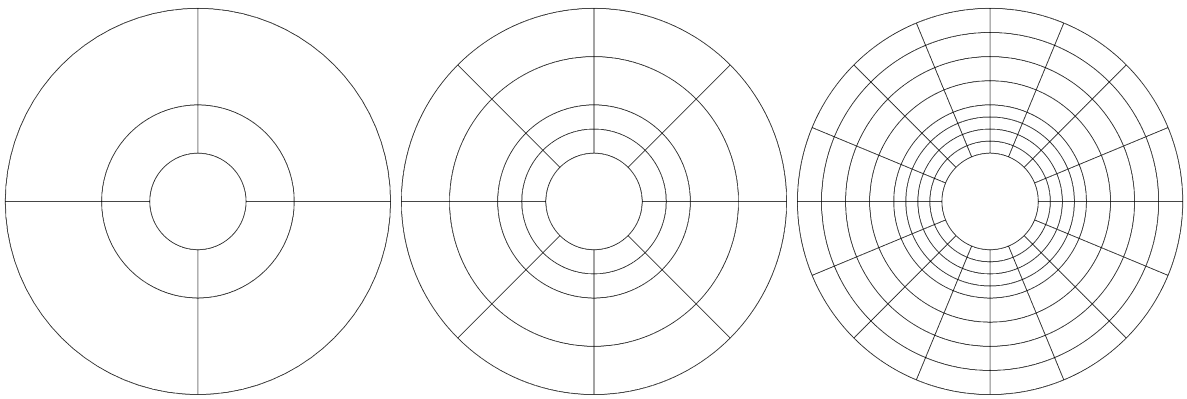


Fig. 3. A sequence of curved quadrilateral meshes for the scattering on a circular disc in Concepts.

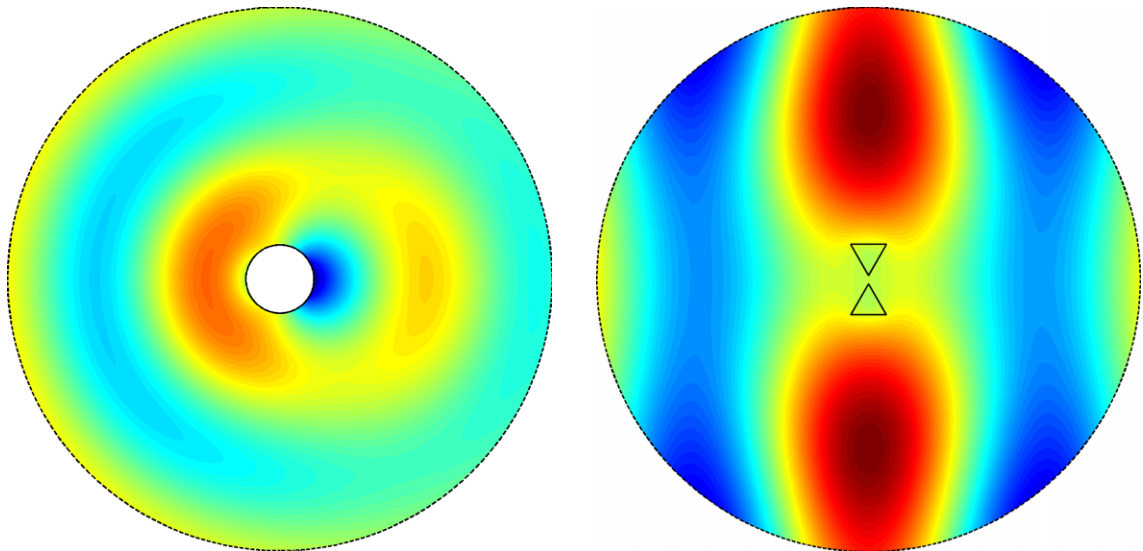


Fig. 4. The scattered field (real part) for the model problem A (left) and model problem B (right), both with $R = 8$.

with a polynomial degree which is high enough so that the discretization error of the reference solution can be neglected. The discretization error is computed as the difference of the discrete solution and the reference solution.

The results of the convergence analysis are shown in Fig. 6. The observed convergence orders of the discretization error in the $H^1(\Omega)$ -seminorm are 1.0 for $p = 1$, 2.0 for $p = 2$ and 3.0 for $p = 3$, and in the $L^2(\Omega)$ -norm 2.0 for $p = 1$, 3.0 for $p = 2$ and 4.0 for $p = 3$. Hence, the convergence orders meet the orders of the best-approximation error. In the variational formulation with Feng's conditions integrals of the trace of the solution and its derivatives on the outer boundary Γ are present. Therefore, the convergence of the discretization error on Γ is studied as well. In the $H^1(\Gamma)$ -seminorm the obtained convergence rates are 1.2, 2.0 and 3.0 for $p = 1, 2, 3$, respectively, and correlate to the convergence orders of the best-approximation error. In the $L^2(\Gamma)$ -norm convergence rates of 2.0, 4.0, and 5.3 for $p = 1, 2, 3$, respectively, are observed. These observed convergence rates are for $p = 2$ and $p = 3$ better than those for the best-approximation error of an arbitrary, smooth enough function (see Fig. 5).

Modelling error for model problem B. For model problem B Feng's conditions of different order for a fine mesh with polynomial $p = 6$ are compared. Here, the discretization error is less than $1 \cdot 10^{-6}$ in $L^\infty(\Omega)$ and the modelling error is dominating. In Fig. 8 the modelling error for $R = 8$ using the Feng-0 condition, which is of Robin type, the Feng-2 condition, which is of Wentzell type, and the Feng-4 condition are shown. Increasing the order of the condition leads to a significant error reduction, the error diminishes by a factor of 100 when using Feng-2 instead of Feng-0, and by another factor of 10 when using Feng-4 instead of Feng-2. For the parameters used the Feng-5 conditions do not give a further error reduction. This will only be achieved for larger domain radius R .

Total error for model problem A. Using the proposed finite element method for the scattering using Feng's conditions the discrete solution comprises a discretization error and a modelling error, which in sum can be called the total error. Total error is studied for the model problem A with a fixed domain Ω with $R = 8$, and uniform polynomial degrees $p = 2, 3$. The

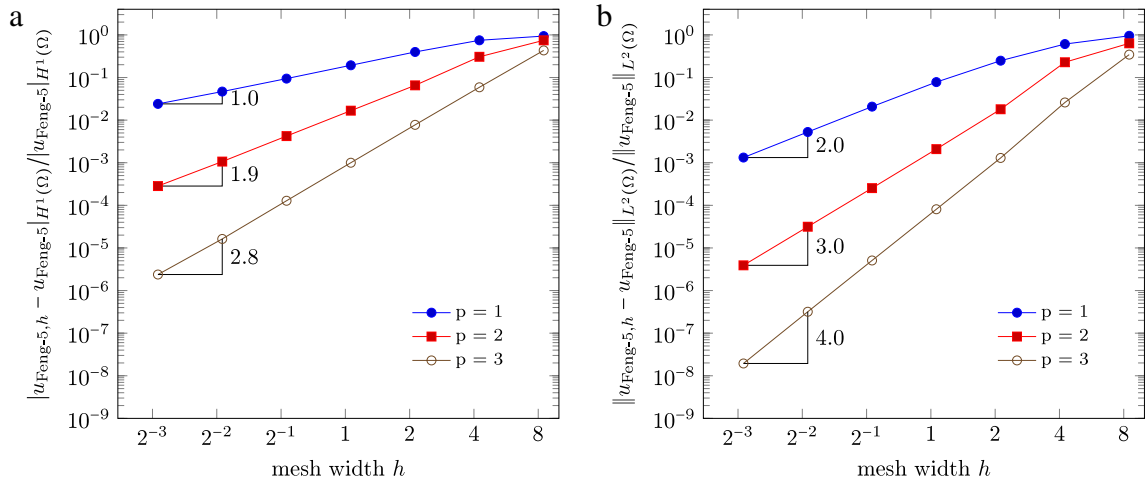


Fig. 5. Convergence of the relative discretization error in (a) the $H^1(\Omega)$ -seminorm, and (b) the $L^2(\Omega)$ -norm for the nonconforming Galerkin formulation with polynomial order $p = 1$ to $p = 3$ for the Feng-5 conditions for the model problem A, where $R_D = 1$ and $R = 8$.

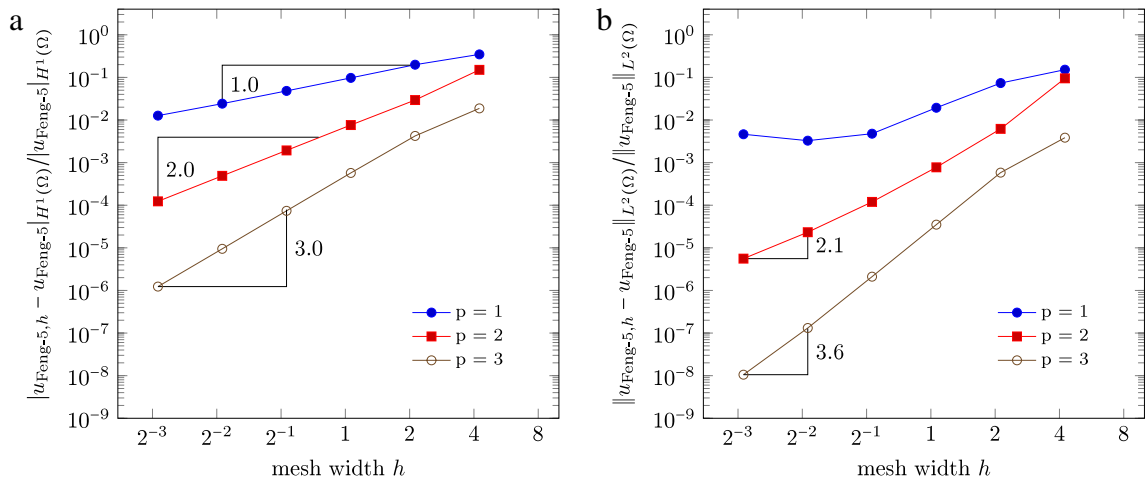


Fig. 6. Convergence of the relative discretization error in (a) the $H^1(\Omega)$ -seminorm, and (b) the $L^2(\Omega)$ -norm for the nonconforming Galerkin formulation with polynomial order $p = 1$ to $p = 3$ for the Feng-5 conditions for the model problem A, where $R_D = 1$ and $R = 3$.

total $L^2(\Omega)$ -error as a function of the mesh-width h for Feng's conditions up to order 5 are shown in Fig. 7. Before reaching the level of the modelling error, the total error decays like $O(h^3)$ or $O(h^4)$ for $p = 2$ or $p = 3$, respectively. In this study, where the domain is fixed with R large enough, the error reduces with mesh refinement due to a decrease of the discretization error and saturates on the level of the modelling error. To obtain a certain level of the total error level the refinement of mesh might not be sufficient, where then either a higher order Feng condition has to be used or the radius of the domain has to be increased.

7. Conclusion

A new finite element method for high-order absorbing boundary conditions was proposed. It combines the classical C^0 -continuous finite element discretization with bilinear forms adopted from discontinuous Galerkin methods for the discretization of the high-order differential operators on the boundary. As there is no need to introduce any auxiliary variable or basis functions with higher regularity, the method is simply integrated into established high-order finite element libraries. In this article, the case of symmetric absorbing boundary conditions in 2D, for which only derivatives of even orders are present, was completely analysed, where we showed that the discretization of the higher-order operators does not hamper the convergence of the finite element method as long as the polynomial degree on the boundary is increased with the order of the boundary condition. More precisely, for each second derivative higher than two the polynomial order on the boundary has to be increased by one, and the same convergence rate as for lower order boundary conditions like the Dirichlet, Neumann, Robin or even Wentzell conditions is obtained. In addition, the changes in the method with additional odd

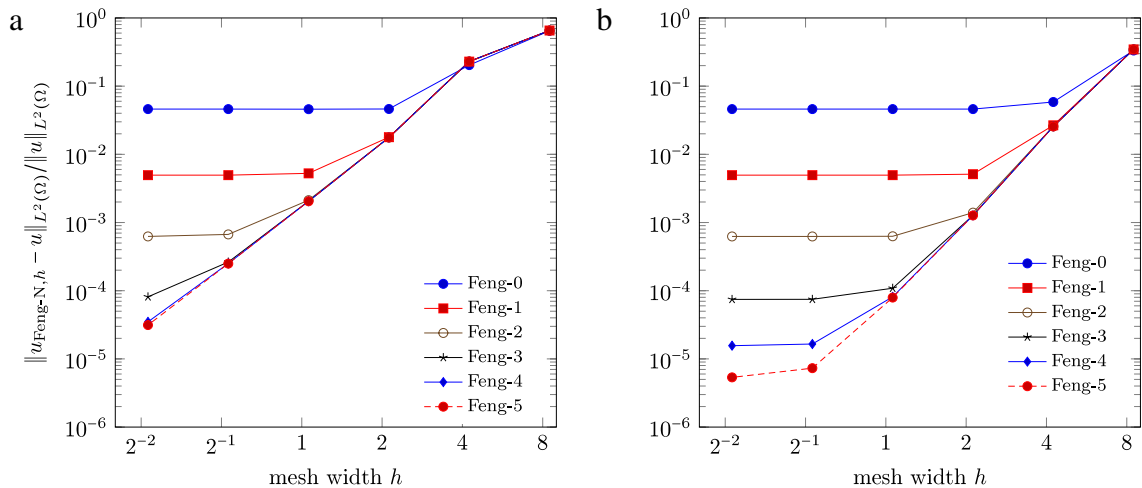


Fig. 7. Convergence of the total error of the finite element discretization for model problem A with Feng's conditions of order 0 to 5 in the mesh-width h for (a) $p = 2$ and (b) $p = 3$. The radius of the domain is $R = 8$.

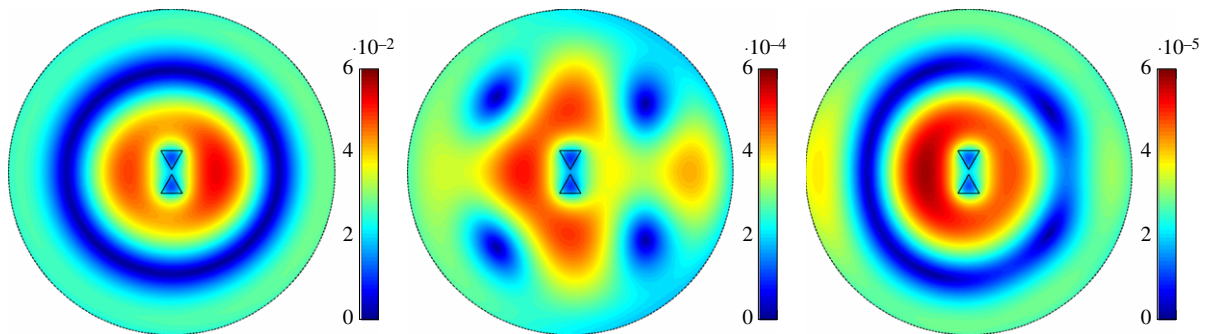


Fig. 8. The error (absolute value) for model problem B with $R = 8$ on a fine mesh resolving the triangles and $p = 6$.

order derivatives in 2D or with powers of Laplace–Beltrami operators in 3D were explained, the latter even for generalized Dirichlet-to-Neumann maps for which the Laplace–Beltrami operator is applied to the Neumann trace. In this way, the method applies to generalized impedance boundary conditions as well as generalized absorbing boundary conditions in 3D such as the BGT conditions. A series of numerical experiments for Feng's absorbing boundary conditions up to order five illustrated the applicability of the method and validates the theoretically obtained estimates of the discretization error.

Based on C^0 -continuous finite element spaces, the proposed interior penalty formulation easily handles local absorbing boundary conditions of arbitrary order, tangential derivatives on curved boundaries of various shapes and with elements of various types and even locally varying polynomial order, both in 2D and 3D. As absorbing boundary conditions of higher order, and hence, higher accuracy, exhibit usually higher order derivatives, the proposed numerical method may lead to significantly higher accuracies with the same discretization in the domain while only increasing the polynomial order in the boundary edges.

As an extension, the method is straightforwardly combined with discontinuous Galerkin methods, where besides the usual terms on the element boundaries additional terms on the domain boundary appear already for second derivatives. Note that in this paper only smooth boundaries and smooth coefficients α_j were considered. An extension of the proposed method to boundaries with a corner or to the case where the coefficients α_j are only piecewise smooth is not a trivial task and has to be studied separately.

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